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Thermodynamics with an Action Principle (2nd edition)

Heat and gravitation

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ABSTRACT Some features of hydro- and thermodynamics, as applied to atmospheres and to stellar structures, are puzzling: 1. The suggestion, first made by Laplace, that our atmosphere has an adiabatic temperature distribution, is confirmed for the lower layers, but the reason why it should be so is understood only qualitatively. 2. Arguments in which a concept of energy plays a role, in the context of hydro-thermodynamical systems and gravitation, are often flawed, and some familiar results concerning the stability of model stellar structures, first advanced at the end of the 19th century and repeated in the most modern textbooks, are less than completely convincing. 3. The standard treatment of relativistic thermodynamics does not allow for a systematic treatment of mixtures, such as the mixture of a perfect gas with radiation. 4. The concept of mass in applications of general relativity to stellar structure is unsatisfactory. It is proposed that a formulation of thermodynamics as an action principle may be a suitable approach to adopt for a new investigation of these matters.

We formulate thermodynamics of ideal gases in terms of an action principle and study the interaction between an ideal gas and the photon gas, or heat. The action principle provides a hamiltonian functional, not available in traditional approaches where familiar expressions for the energy have no operative meaning. The usual polytropic atmosphere in an external gravitational field is examined, in order to determine to what extent it is shaped by radiation. It is easy to understand that radiation sustains the atmosphere (prevents cooling), but the temperature profile is largely determined by intrinsic properties of the gas and it is difficult to interpret it as an effect of radiation. This has led some people to question whether the rule of uniform temperature as an absolute condition for equilibrium is valid in the presence of a gravitational field. An experiment that involves a centrifuge and that has wider implications in view of the equivalence principle, is proposed, to ascertain the influence of gravitation on the equilibrium distribution with a very high degree of precision. A new formulation of the concept of radiative equilibrium is proposed.

The choice of boundary conditions for radial, stellar stability calculations is clarified with the help of a properly defined, conserved mass distribution.

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I. Introduction

The statement that any two thermodynamic systems, each in a state of equilibrium with a well defined temperature, in thermal equilibrium with each other, must have the same temperature, is a central tenet of thermodynamics. A natural extension is that the temperature in an extended but closed system in a state of equilibrium must be uniform, but there does not seem to be universal agreement on whether this is true in the presence of gravitational fields. The question comes up in the investigation of terrestrial or stellar atmospheres, where the gravitational forces create a non-uniform density distribution.

This paper is a study of the polytropic atmosphere. We want to know if (or to what extent) the polytropic relations are to be attributed to intrinsic properties of the gas, or to radiation. In this we are guided by a strong preference for action principles.

The textbooks present hydrodynamics as the theory of a continuous distribution of matter, described in the simplest case by two fields or distributions: a density field and a velocity field, both defined over \mathbb{R}^3 or a portion thereof. The role of temperature is often disguised, assumed to be determined by the density and the pressure. Classical thermodynamics, on the other hand, is primarily the study of states of equilibrium, with uniform density and temperature, and relations between such states. In this context extremum principles play a pivotal role; see for example Callen (1960). Texts that deal with flow of matter or with temperatures that vary in time or from one point to another are found under the heading of heat transfer, fluctuations or thermodynamics of irreversible processes. See for example Stanyukovich (1960), Müller (2007). These studies rely heavily on conservation laws, but variational principles are rarely mentioned. Texts that most closely parallel the present work are found under radiation hydrodynamics (Castor 2004).

In this introduction we study a simple system from the point of view of hydrodynamics, on the basis of a well known action principle. The concept of temperature appears, but not as a dynamical variable. We offer a brief review of the history of the polytropic atmosphere (Section I.6) and stress the role of mass. In Section II, we extend the action principle to include the temperature as an independent field variable. A lagrangian describes the configurations on a single polytrope. The Euler-Lagrange equations include the gas law as well as the polytropic relation between pressure and density. The familiar expression for the internal energy of an ideal, polytropic gas coincides with the hamiltonian; this appears to be a significant result. The energy and the pressure of radiation are incorporated in a natural way (Section II.6).

The theory is mathematically complete in the sense that no additional input from underlying microscopic physics is needed; as an example we derive a virial theorem that is proper to the action principle (Section II.7). But it is physically incomplete since the role of convection and heat flow is not explicitly accounted for.

To contribute to the debate on the question of whether an isolated atmosphere in a gravitational field tends to isothermal equilibrium we study an ideal gas in a centrifuge and invoke the equivalence principle (Section II.8).

In Section III we study the effect of sources of heat that induce transitions between polytropes. The radiation field contributes as a source of entropy. The principal idea behind this work was the hope of understanding, in a quantitative way, why real atmospheres tend to be polytropic; in this we have had only limited success.

In Section IV we take up the problem of the stability of polytropic, atmospheric models. A principal advantage of the method is that it provides us with a hamiltonian, expressed in terms of the dynamical variables. Some classical stability studies are found wanting, because of *ad hoc* definitions of various energies, and inappropriate boundary conditions.

Section V makes the passage to General Relativity. Section VI has a summary of conclusions and several proposals for additional work, theoretical as well as experimental.

I.1. Hydrodynamics

The textbook introduction to hydrodynamics deals with a density field ρ and a velocity field \vec{v} over \mathbb{R}^3 , subject to two fundamental equations, the equation of continuity,

$$\dot{\rho} + \text{div}(\rho \vec{v}) = 0, \quad \dot{\rho} := \frac{\partial \rho}{\partial t}, \quad (1.1)$$

and the hydrodynamical equation (Bernoulli 1738)

$$-\text{grad } p = \rho \frac{D}{Dt} \vec{v} := \rho (\dot{\vec{v}} + \vec{v} \cdot \text{grad } \vec{v}). \quad (1.2)$$

This involves another field, the scalar field p , interpreted as the local pressure. The theory is incomplete and requires an additional equation relating p to ρ . It is always assumed that this relation is local, giving $p(x)$ in terms of the density at the same point x , and instantaneous.

I.2. Laminar flow

Since we are reluctant to take on difficult problems of turbulence, we shall assume, here and throughout, that the velocity field can be represented as the gradient of a scalar field,

$$\vec{v} = -\text{grad } \Phi. \quad (1.3)$$

In this case the hydrodynamical condition is reduced to

$$\text{grad } p = \rho \text{ grad } (\dot{\Phi} - \vec{v}^2/2). \quad (1.4)$$

To complete this system one needs a relation between the fields p and ρ .

Assume that there is a local functional $V[\rho]$ such that

$$p = \rho V' - V, \quad V' := dV/d\rho. \quad (1.5)$$

In this case $dp = \rho dV'$ and the equation becomes, if $\rho \neq 0$,

$$\text{grad } V' = \text{grad } (\dot{\Phi} - \vec{v}^2/2) \quad (1.6)$$

or

$$V' = \dot{\Phi} - \vec{v}^2/2 + \lambda, \quad \lambda \text{ constant.} \quad (1.7)$$

The potential $V[\rho]$ is defined by p modulo a linear term, so that the appearance of an arbitrary constant is natural. It will serve as a Lagrange multiplier.

The introduction of a velocity potential guarantees the existence of a first integral of the motion, a conserved energy functional that will play an important role in the theory.

I.3. Variational formulation

Having restricted our scope, to account for laminar flows only, we have reduced the fundamental equations of simple hydrodynamics to the following two equations,

$$\begin{aligned}\partial_\mu J^\mu &= 0, \quad J^t := \rho, \quad \vec{J} := \rho \vec{v}, \\ V' &= \dot{\Phi} - \vec{v}^2/2 + \lambda,\end{aligned}\tag{1.8}$$

together with the defining equations

$$\vec{v} = -\text{grad } \Phi, \quad p := \rho V' - V.\tag{1.9}$$

It is well known that these equations are the Euler-Lagrange equations associated with the action (Fetter and Walecka 1980)

$$A = \int dt d^3x \mathcal{L}, \quad \mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 + \lambda) - V[\rho].\tag{1.10}$$

The value of this last circumstance lies in the fact that the variational principle is a much better starting point for generalizations, including the incorporation of symmetries, of special relativity, and the inclusion of electromagnetic and gravitational interactions. It also gives us a valid concept of a total energy functional.

I.4. On shell relations

The action (1.10) contains only the fields Φ and ρ , and the Euler-Lagrange equations define a complete dynamical framework, but only after specification of the functional $V[\rho]$. The pressure was defined by Eq.(1.9), $p := \rho V' - V$, and one easily verifies that, on the trajectory, by virtue of the equations of motion,

$$p = \mathcal{L} \quad (\text{on shell}).\tag{1.11}$$

This fact has been noted, and has led to the suggestion that the action principle reduce to the minimization of $\int p$ with respect to variations of p defined by thermodynamics (Taub 1954), (Bardeen 1970), (Schutz 1970). But more is needed, including an off shell action. After adopting the action (1.10) it remains to relate the choice of the potential V to the thermodynamical properties of the fluid.

I.5. Equation of state and equation of change

An ideal gas at equilibrium, with constant temperature, obeys the gas law

$$p/\rho = \mathcal{R}T.\tag{1.12}$$

Pressure and density are in cgs units and

$$\mathcal{R} = (1/\mu) \times .8314 \times 10^8 \text{ erg/K},$$

where μ is the atomic weight. In this paper the validity of the gas law is assumed to hold, locally at each point of the gas, under all circumstances, including the case that gravitational and electromagnetic fields are present. Effective values of μ are

$$\text{Atomic hydrogen : } \mu = 1, \quad \text{Air : } \mu = 29, \quad \text{Sun : } \mu = 2.$$

Equation (1.12) is the only equation that is referred to as an ‘equation of state’. Other relations, to be introduced next, are ‘equations of change’, this term taken from Emden’s “Zustandsänderung”, for their meaning is of an entirely different sort. The most important is the polytropic relation

$$p = A\rho^{\gamma'}, \quad A, \gamma' \text{ constant.} \quad (1.13)$$

This relation defines a polytropic path or polytrope in the p, v diagram ($v = 1/\rho$). A polytropic atmosphere is one in which, as one moves through the gas, the variables ρ and p change so as to remain always on the same polytrope, the temperature being determined by Eq.(1.12) always. Eq.(1.13) is a statement about the system, not about the gas *per se*.

The index of the polytrope is the positive number n defined by

$$\gamma' =: 1 + \frac{1}{n'}.$$

Important special cases are

$$\begin{aligned} n' &= 0, \quad \gamma = \infty, \quad \rho = \text{constant}, \\ \gamma' &= C_P/C_V, \quad \text{specific entropy} = \text{constant}, \\ n' &= \infty, \quad \gamma' = 1, \quad T = \text{constant}. \end{aligned}$$

The numbers γ , n are defined by

$$\gamma := C_P/C_V =: 1 + \frac{1}{n}.$$

The number n is the adiabatic index of the gas. According to statistical mechanics $2n$ is the number of degrees of freedom of each molecule in the gas. That atmospheres tend to be polytropic is an empirical fact.

The case that $\gamma' = \gamma$ is of a special significance. A polytrope with $\gamma' = \gamma$ is a path of constant specific entropy; changes along such polytropes are reversible and adiabatic; these polytropes and no others are adiabats.

Fix the constants A, γ' and consider an associated stationary, polytropic atmosphere. Since both (1.12) and (1.13) hold we have

$$p = \text{const.} \cdot \rho^{\gamma'}, \quad p = \text{const.} \cdot T^{\frac{\gamma'}{\gamma'-1}}, \quad T = \text{const.} \cdot p^{1-1/\gamma'}. \quad (1.14)$$

In any displacement along a polytrope from a point with pressure p and temperature T , we shall have $d\rho/\rho = (1/\gamma')dp/p$, so that an increase in pressure leads to an increase in

density that is greater for a smaller value of γ' . If a parcel of gas in this atmosphere is pushed down to a region of higher pressure, by a reversible process, then it will adjust to the ambient pressure. If $\gamma > \gamma'$, then it will acquire a density that is lower than the environment; it will then rise back up; this atmosphere is stable. But if $\gamma' > \gamma$ then the parcel will be denser than the environment and it will sink further; this atmosphere is unstable. Thus we have:

A stable, polytropic atmosphere must have $\gamma' < \gamma$, $n' > n$.

Most stable is the isothermal atmosphere, $\gamma' = 1$.

In hydrodynamics, the isothermal atmosphere can be given a lagrangian treatment by taking

$$V = \mathcal{R}T\rho \log \rho. \quad (1.15)$$

We suppose that the gas is confined to the section $z_0 < z < z_0 + h$ of a vertical cylinder with base area \mathcal{A} and expect the density to fall off at higher altitudes. A plausible action density, for a perfect gas at constant temperature T in a constant gravitational field $\phi = gz$, g constant, is

$$\mathcal{L}[\Phi, \rho] = \rho(\dot{\Phi} - \vec{v}^2/2 - gz + \lambda) - \mathcal{R}T\rho \log \rho. \quad (1.16)$$

We may consider this an isolated system with fixed mass and fixed extension.

At equilibrium $\dot{\Phi} = 0$, $\vec{v} = 0$, $\dot{\rho} = 0$ and the equation of motion is $V' = \mathcal{R}T(1 + \log \rho) = \lambda - gz$, hence

$$\rho(x, y, z) = e^{-1+\lambda/\mathcal{R}T} e^{-gz/\mathcal{R}T}, \quad M = \mathcal{A} \frac{\mathcal{R}T}{g} e^{-1+\lambda/\mathcal{R}T} (1 - e^{-gh/\mathcal{R}T}) e^{-gz_0/\mathcal{R}T}$$

and after elimination of λ

$$\rho = \frac{gM}{\mathcal{A}\mathcal{R}T} \frac{e^{-g(z-z_0)/\mathcal{R}T}}{1 - e^{-gh/\mathcal{R}T}}, \quad p = \frac{gM}{\mathcal{A}} \frac{e^{-g(z-z_0)/\mathcal{R}T}}{1 - e^{-gh/\mathcal{R}T}}. \quad (1.17)$$

There is no difficulty in taking the limit $h \rightarrow \infty$. The volume becomes infinite but it can be replaced as a variable by the parameter z_0 . This atmosphere is stable; a proof is presented in Section IV.1.

The isothermal atmosphere is usually abandoned in favor of the polytropic atmosphere.

A polytropic gas can be described by the lagrangian (1.10), with

$$V = \hat{a}\rho^{\gamma'}, \quad \hat{a}, \gamma' \text{ constant.}$$

Variation with respect to ρ gives

$$p = \frac{\hat{a}}{n'} \rho^{\gamma'}, \quad \frac{1}{n'} = \gamma' - 1,$$

to be supplemented by the gas law, Eq.(I.12). Among the many applications the following are perhaps the most important. In the case of sound propagation the gas is initially awakened from equilibrium turpor and then left in an isolated, frenzied state of oscillating density and pressure, with the temperature keeping pace in obedience to the gas law. All three of the relations (1.14) are believed to hold, with $\gamma' = \gamma$. The oscillations are usually too rapid for the heat to disseminate and equalize the temperature, so that the neglect of heat transfer may be justified. In applications to the atmospheres one uses the polytropic equation of change (1.13) and obtains the temperature from the gas law. Understanding the resultant temperature gradient in terms of convection, or as the effect of the heating of the air by solar radiation, or both, is one of the main issues on which we had hoped to gain some understanding.

At mechanical equilibrium $\vec{v} = 0$, $\dot{\rho} = 0$ and $\lambda - gz = \hat{a}\gamma\rho^{1/n}$, hence

$$\rho = \left(\frac{\lambda - gz}{\hat{a}\gamma}\right)^n.$$

Since the density must be positive one does not fix the volume but assumes that the atmosphere ends at the point $z_1 = \lambda/g$. Then

$$M = \mathcal{A}\left(\frac{g}{\hat{a}\gamma}\right)^n \int_{z_0}^{z_1} (z_1 - z)^n dz = \frac{\mathcal{A}h}{n+1} \left(\frac{gh}{\hat{a}\gamma}\right)^n.$$

This fixes h and thus z_1 and λ . If the atmosphere is an ideal gas then the temperature varies with altitude according to

$$\mathcal{R}T = p/\rho = \frac{\hat{a}}{n}\rho^{1/n} = g\frac{z_1 - z}{n+1} \quad (1.18)$$

Because the lagrangian does not contain T as a dynamical variable it is possible to impose this condition by hand.

One would not apply this theory down to the absolute zero of temperature, but even without going to such extremes it seems risky to be predicting the temperature of the atmosphere without having made any explicit assumptions about the absorption or generation of heat that is said to be required to sustain it. Yet this has been the basis for the phenomenology of stellar structure, as well as the earth's atmosphere, from the beginning (e.g. Lane 1870, Ritter 1878).

The success of the isotropic model is notorious, and this success can be explained in physical terms, but the theory is incomplete since it does not account for heat flow, nor convection, both of which are needed to complete the picture.

For air, with atomic weight 29, $\mathcal{R} = 2.87 \times 10^6 \text{ ergs/gK}$ and $n = 2.5$. At sea level, $g = 980 \text{ cm/sec}^2$, the density is $\rho = 1.2 \times 10^{-3} \text{ g/cm}^3$, the pressure $p = 1.013 \times 10^6 \text{ dyn/cm}^2$. Thus

$$p/\rho = .844 \times 10^9 \text{ cm}^2/\text{sec}^2, \quad T = T_0 = 294\text{K}, \quad z_1 = 3.014 \times 10^6 \text{ cm} \approx 30\text{km}.$$

and the dry lapse rate at low altitudes is $-T' = 294/z_1 = 9.75\text{K/km}$. The opacity that is implied by this is mainly due to the presence of CO_2 in the atmosphere. Humidity increases the opacity and decreases the lapse rate by as much as a factor of 2.

I.6. Historical notes on polytropic atmospheres

Observations of reversible transformations of near-ideal gases, carried out during the 19th century, can be summarized in what is sometimes called the laws of Poisson,

$$\rho \propto T^{n'}, \quad p \propto T^{n'+1}, \quad p \propto \rho^{\gamma'}, \quad \gamma' = 1 + \frac{1}{n'} \text{ constant.}$$

In the original context all the variables are constant and uniform. The exponents as well as the coefficients of proportionality are the same for all states that are related by reversible transformations. Statistical mechanics explained this result and confirmed the experimental value $\gamma' = \gamma = C_P/C_V$. As far as can be ascertained, the presence of terrestrial gravitation and ambient radiation had no effect on these experiments. In a first extrapolation the same relations were taken to hold locally in dynamic situations, as in the case of sound propagation. The gas is not in equilibrium and the variation of the temperature from point to point, and with time, is obtained from the gas law.

For the atmosphere of the earth it was at first proposed that the temperature would be uniform. However, the existence of a temperature gradient was soon accepted as an incontrovertible experimental fact. The first recorded recognition of this, together with an attempt at explaining the same, may be that of Carnot, in the paper in which he created the science of thermodynamics (Carnot 1824). Carnot quotes Laplace: “N’est-ce pas au refroidissement de l’air par la dilatation qu’il faut attribuer le froid des régions superieures de l’atmosphere? Les raisons données jusqu’ici pour expliquer ce froid sont tout a fait insuffisantes; on dit que l’air des régions elevées, recevant peu de chaleur reflechie par la terre, et rayonnant lui meme vers les espaces celestes, devait perdre de calorique, et que c’était lá la cause de son refroidissement; ... ” This may be the first time that the influence of radiation is invoked. The temperature gradient is attributed to the greenhouse effect, and Laplace was an early skeptic, for he continues “...mais cette explication ce trouve detruite si l’on remarque qu’a égale hauteur le froid regne aussi bien et meme avec plus d’intensité sur les plaines elevées que sur les sommets des montagnes ou que dans les parties d’atmosphere éloignées du sol.” It is not clear that the two explanations are at odds with each other; Laplace apparently postulates that the atmospheres over lands at different elevations are related by adiabatic transformations, without explaining why.

By rejecting the role of radiation as the cause of the temperature gradient, Laplace seems to suggest that the same would be observed in an atmosphere subject to gravitation but totally isolated from radiation, neither exposed to the radiation coming from the sun nor radiating outwards. As was strongly emphasized in the later phases of this debate, this would contradict the belief that the thermal equilibrium of any isolated system, gravitation and other external forces notwithstanding, is characterized by a uniform temperature.

In 1862 W. Thomson, in the paper “On the convective equilibrium of the temperature in the atmosphere”, defines convective equilibrium with these words “When all parts of a fluid are freely interchanged and not sensibly influenced by radiation and conduction, the temperature is said to be in a state of convective equilibrium.” He then goes on to say that an atmosphere that is in convective equilibrium is a polytrope, and we think that he means an adiabat, since this is probably implied by the words “freely interchanged”, although the value of γ is taken from experiment and not from statistical mechanics. At first sight

the clause “and not sensibly influenced by radiation” would seem to imply that an isolated atmosphere has a temperature gradient, but this conclusion would be premature, as we shall see.

In 1870 H.J. Lane made the bold assertion that the laws of Poisson may be satisfied in the Sun. The terrestrial atmosphere (or part of it) had already been found to be well represented by the same relations. Referring to Lane’s paper Thomson, now lord Kelvin, explains how convective equilibrium comes about (Thomson 1907). He argues that the atmosphere is not, cannot be, at rest, and this time radiation plays an essential role. The upper layers loose heat by radiation and the lower temperature leads to an increase in density. This produces a downward current that mixes with a compensating upward drift of warmer air. This continuing mixing takes place on a time scale that is too short for adjacent currents to exchange a significant amount of heat by conduction or radiation, especially since the variations of temperature are very small. It is evident that Thomson offers his explanation of the temperature gradient to account for its absence in an isolated atmosphere, for he says that, “an ideal atmosphere, perfectly isolated from absorption as well as emission of radiation, will, after enough time has passed, reach a state of uniform temperature, irrespective of the presence of the gravitational field”. Thomson accepts the mechanism of Laplace and Carnot, as it is at work in the real atmosphere, but he goes further. He believes that the lower temperature aloft is intimately tied to the existence of radiation, implying that it is driven by net outwards radiation. (The effect of solar radiation on the terrestrial atmosphere is not explicitly mentioned.) It is difficult to tell whether or not Thomson is in disagreement with Laplace, but the precision of his statements represents a marked improvement over his predecessors and his earlier work.

The principal developers of the field, Ritter (1878-1883) and Emden (1907), seem to accept the idea of convective equilibrium. It may be pointed out, however, that this mechanism is in no way expressed by the equations that these and other authors use to predict the behaviour of real atmospheres. *The concept of convective equilibrium is introduced to one purpose only: to avoid contradiction with the ideas on thermal equilibrium of isolated systems. It receives no quantitative theoretical treatment.*

Nor was it accepted by everybody. A famous incidence involves Loschmidt (1876), who believed that an isolated atmosphere, at equilibrium in a gravitational field, would have a temperature gradient. But arguments presented by Maxwell and Boltzmann (1896) led Loschmidt to withdraw his objections, which is hardly surprising given the authority of these two. Nevertheless, it may be pointed out that no attempt was made, to our knowledge, to settle the question experimentally. See however (Graeff 2008).

An alternative to convective equilibrium was proposed by Schwarzschild (1906) and critically examined by Emden. To understand how it works we turn to Emden’s book of 1907, beginning on page 320. Here he invokes a concept that is conspicuously absent from all his calculations on polytropic spheres in the rest of the book: heat flow. It must be agreed that the atmosphere is not completely transparent, and that heat flow is an inevitable consequence of the existence of a temperature gradient. *The most important observation is that heat flow is possible in stationary configurations ($\dot{T} = 0$) provided that the temperature gradient is constant.* The heat flux due to conduction and radiation can

be expressed as

$$\vec{F} = C \vec{\nabla} T, \quad F^i = C^{ij} \partial_j T,$$

where the tensor C includes the thermal conductivity as well as the effective coefficient of heat transfer by radiation. The divergence of the flux is the time rate of change of the temperature due to conduction and radiation. In a stationary, terrestrial atmosphere, with no local energy creation, this must vanish. Emden's atmospheres are polytropes, with temperature gradients that are constant (it appears that he takes C to be constant). That is interesting, for it reminds us that the entire edifice implicitly demands that this condition, of a constant temperature gradient, be satisfied.

We note that the direction of flow is from hot to cold, outwards. Confining ourselves to planetary atmospheres, with no local energy generation, this calls for an explanation, since the ultimate source of energy is above. Here we have to return to the oldest explanation of the existence of a temperature gradient, dismissed by Laplace (*op. cit.*): the greenhouse effect. The atmosphere is highly transparent to the (high frequency) radiation from the Sun but opaque to the thermal radiation to which it is converted by the ground. The atmosphere is thus heated from below!

If the atmosphere is stable in the sense discussed above, when $\gamma' < C_P/C_V$, then it is not necessary to assume that any convection takes place. In this case one speaks of (stable) 'radiative equilibrium'. Convective equilibrium steps in when the stationary atmosphere is unstable, but it is no longer used to explain the existence of a temperature gradient; it is the effect rather than the cause of it.

A difficulty is present in all accounts of stellar structure up to 1920. The energy observed to be emitted by the Sun, attributed to contraction of the mass and the concomitant release of internal energy, was far too small to account for the age of the sun as indicated by the geological record. The situation changed with the discovery of thermonuclear energy generation. Now there is plenty of energy available. At the same time there arose the realization that convection sometimes plays a very modest role; the concept of convective equilibrium was put aside and with it, Kelvin's explanation of the temperature gradient. According to Eddington (1926), "convective equilibrium" must be replaced by "radiative equilibrium". He does not claim that this new concept explains the temperature gradient as well as Kelvin's convective equilibrium does, but in fact the local generation of heat by thermonuclear processes creates an outward flow of heat and a negative temperature gradient.

Emden's implicit invocation of the heat equation reminds us that this equation probably should replace the simple rule that 'a system in equilibrium must have a uniform temperature'.

Finally, there emerges a physical picture that seems to account for all the principal features of some atmospheres, in a qualitative way. If one begins with an isolated atmosphere in equilibrium one may take it as an axiom that the temperature is uniform. A weak dose of radiation upsets the static equilibrium and the gas goes into a state of convective equilibrium, characterized by an adiabatic temperature profile. When the intensity of the radiation increases it produces an outward heat flux and with it an added pressure. This pressure is the gradient of the energy of radiation and the additional 'potential' has to be overcome when a parcel of gas is moving downward; that is, the polytropic index

is increased. This in turn stabilizes the atmosphere against convection and eventually it reaches a state of ‘radiative equilibrium’, with insignificant convection. See (Cox and Giuli 1968), page 271. If this interpretation is correct then the theory must be completed by inclusion of heat flow as an additional degree of freedom, and by an account of the properties of the coefficient C . A more distant goal would be the inclusion of convection into the mathematical description.

I.7. The mass

To speak of a definite, isolated physical system we must fix some attributes, and among such defining properties we shall include the mass. We insist on this as it shall turn out to be crucial to the logical coherence of the theory. The density ρ will be taken to have the interpretation of mass density, and the total mass is the constant of the motion

$$M = \int d^3x \rho.$$

Such integrals, with no limits indicated, are over the domain Σ of definition of ρ and is the total extension of our system in \mathbb{R}^3 .

Since the total mass is a constant of the motion it is natural to fix it in advance and to vary the action subject to the constraint $\int_{\Sigma} d^3x \rho(x) = M$. We introduce a Lagrange multiplier and the action takes the form

$$A = \int_{\Sigma} d^3x \left(\rho(\dot{\Phi} - \vec{v}^2/2) - V \right) + \lambda \left(\int_{\Sigma} d^3x \rho - M \right). \quad (1.19)$$

In the simplest case of a polytropic equation of state and no external forces we get the following equations of motion

$$\partial_{\mu} J^{\mu} = 0, \quad V' = \dot{\Phi} - \vec{v}^2/2 + \lambda = \hat{a}\gamma\rho^{1/n}.$$

Here λ is to be chosen for each solution so as to satisfy the constraint. In the case of a static solution with $\dot{\Phi} = 0, \vec{v} = 0$ the density is constant. Assuming a finite system with volume \mathcal{V} we have $M = \rho\mathcal{V} = (\lambda/\hat{a}\gamma)^n\mathcal{V}$ and since M is given,

$$\rho = \frac{M}{\mathcal{V}}, \quad p = \frac{\hat{a}}{n} \left(\frac{M}{\mathcal{V}} \right)^{\gamma}, \quad \lambda = \hat{a}\gamma \left(\frac{M}{\mathcal{V}} \right)^{1/n}. \quad (1.20)$$

The conservation of mass has important implications for boundary conditions.

II. The first law

II.1. Thermodynamic equilibrium

A state of thermodynamical equilibrium of a system that consists of a very large number of identical particles is defined by the values of 3 variables, *a priori* independent, the density D , the pressure P and the temperature T . These are variables taking real values; they apply to the system as a whole. In the case of any particular system there is one relation that holds for all equilibrium states, of the form

$$T = f(D, P).$$

It is written in this form, rather than $F(T, D, P) = 0$, because a unique value of T is needed to define a state of equilibrium between two systems that are in thermal contact with each other: it is necessary and sufficient that they have the same temperature. This statement incorporates the zeroth law.

If we divide our system into subsystems then these will be in thermal equilibrium with each other only if they have the same temperature. This, at least, is the inherited wisdom; we shall honor it as long as possible.

The ideal gas at equilibrium is defined by global variables T, D, P , and two relations. The principal one is the gas law

$$P/D = \mathcal{R}T, \quad \mathcal{R} = .8314 \times 10^8 \text{ ergs}/K,$$

where $1/D$ is the volume of a mole of gas. The other may take the form of an expression for the internal energy.

II.2. The ideal gas in statistical mechanics

Here again we consider a gas that consists of identical particles (Boltzmann statistics), each with mass m and subject to no forces. It is assumed that the i th particle has momentum \vec{p}_i and kinetic energy $\vec{p}_i^2/2m$. This is an ideal gas, satisfying the relation $P/D = \mathcal{R}T$ at equilibrium. It is assumed that the number N of particles with energy E is given by the Maxwell distribution

$$N(E) \propto e^{-E/kT}, \tag{2.1}$$

which implies a constant density in configuration space. Now place this gas in a constant gravitational field, with potential $\phi(x, y, z) = gz$, g constant. Since the potential varies extremely slowly on the atomic scale it is plausible that, at equilibrium, each horizontal layer (ϕ constant) is characterized by a constant value of the temperature, density and pressure. Since neighbouring layers are in thermal contact with each other the temperature must (?) be the same throughout,

$$T(z) = T = \text{constant},$$

and

$$p(z)/\rho(z) = \mathcal{R}T, \quad z \geq 0. \tag{2.2}$$

The energy of a particle at level z is $\vec{p}^2/2m + mgz$ and (2.1) now implies the following distribution in configuration space,

$$\rho(x, y, z) \propto e^{-mgz/kT}, \quad (2.3)$$

in perfect agreement with (1.17). This supports the appearance of the logarithm in the expression for the potential, Eq.(1.15). Both derivations of the distribution rest on the assumption that the temperature is constant throughout the system.

We conclude that the static solutions of the action principle, with action density (1.16) and T fixed, describe the equilibrium states of an ideal gas at fixed temperature T in the sense of thermodynamics and statistical mechanics, even in the presence of the gravitational field, when no account is taken of radiation. But we do not know under what conditions the temperature will actually be uniform.

About this question there has been some debate, see e.g. Waldram (1985), page 151. It is said that the kinetic energy of each atom in a monatomic gas is $3kT/2$ and that, when the temperature is the same everywhere, this is paradoxical because it does not take account of the potential energy of the atom in the gravitational field. The incident involving Loschmidt, Maxwell and Boltzmann has already been mentioned.

II.3. The first law and the internal energy

Is further generalization possible? Can we extend the model to the case that the temperature varies with time? The action must be modified, for the temperature becomes a dynamical field. Is the temperature one of the variables with respect to which the action must be minimized? We need an equation of motion to predict its evolution. (The heat equation will be discussed later.) The usual approach is to lay down the additional equation by fiat (Section 1.5); is this completely satisfactory? Would it perhaps be preferable to have it appear as the result of minimizing the action with respect to variations of the temperature field?

To prepare for the generalization we shall examine some of the main tenets of thermodynamics in the context of the action principle. The question of whether or not it is profitable to treat the temperature as a dynamical field variable in the context of the action principle can best be assessed a little later (Section III.3).

Suppose that the system is in thermal and mechanical isolation except for a force that is applied to the boundary. The system is in an equilibrium state with temperature T . The applied force is needed to hold the gas within the boundary of the domain Σ , then decreased by a very small amount leading to a displacement of the boundary and an increase of the volume by a small amount $d\mathcal{V}$. Assume that this process is reversible. The work done by the applied force is

$$dW = -pd\mathcal{V}. \quad (2.4)$$

The first law states that, if the system is in thermal isolation, then this quantity is the differential of a function $U(T, \mathcal{V})$ that is referred to as the internal energy of the system.

Consider the system that consists of an ideal gas confined to a volume \mathcal{V} and experiencing no external forces, not even gravitation. If the gas expands at constant pressure

the work done by the gas is $pd\mathcal{V}$ and Eq.(1.12) tells us that,

$$pd\mathcal{V} = \mathcal{R}TM \frac{d\mathcal{V}}{\mathcal{V}}. \quad (2.5)$$

The idea of energy conservation suggests a concept of “internal energy”. It is assumed that, under certain circumstances, the work done by the gas is at the expense of an internal energy U so that

$$pd\mathcal{V} + dU = 0,$$

or

$$\mathcal{R}TMd\mathcal{V}/\mathcal{V} + dU = 0.$$

It is an experimental fact (Gay-Lussac 1827, Joule 1850) that the internal energy of an ideal gas is independent of the volume (see below) and the more precise statement that the internal energy density u is proportional to $\mathcal{R}T\rho$ is often included in the definition of the ideal gas (Finkelstein 1969, page 7). Thus

$$u = \hat{c}_V \mathcal{R}T\rho, \quad U = \hat{c}_V \mathcal{R}TM.$$

Statistical mechanics gives $\hat{c}_V = n$, where n is the adiabatic index and takes the value $3/2$ for a monatomic gas. Thus $\mathcal{R}TMd\mathcal{V}/\mathcal{V} + dU = \mathcal{R}TMd\mathcal{V}/\mathcal{V} + n\mathcal{R}MdT = 0$, which implies that

$$dT = -\frac{1}{n} \frac{T}{\mathcal{V}} d\mathcal{V}, \quad T \propto \mathcal{V}^{-1/n}. \quad (2.6)$$

This relates the temperature to the volume and replaces the statement that U is independent of the volume. The calculation from (2.4) onward was done with the understanding that $M = \rho\mathcal{V}$ is fixed.

As we see it, the expression for the internal energy in terms of \mathcal{V} and T appears to be somewhat *ad hoc*, derived from external considerations. At the deepest level the concept of energy derives its importance from the fact that it is conserved with the passage of time, by virtue of the dynamics. The defining equations of hydrodynamics do not admit a first integral, and no unique concept of energy; this is a difficulty that our limitation to laminar flow, and the action principle, will allow us to overcome. In modern versions of thermodynamics, and especially in the thermodynamics of irreversible processes and in radiation thermodynamics, conservation laws are all important, but they are postulated, one by one, not derived from basic axioms as is the case in other branches of physics, and they have a purely formal aspect since they serve only to define various fluxes. See e.g. Stanyukovich (1960), Castor (2004).

II.4. The first law and the hamiltonian

Having adopted an action principle approach we are bound to associate the internal energy with the hamiltonian, but one cannot escape the fact that the hamiltonian density is defined only up to the addition of a constant multiple of the density. When we decide to adopt a particular expression to be used as internal energy over a range of temperatures, we are introducing a new assumption. Any expression for the internal energy, together with the implication that applied forces increase it by an amount determined by the work done, is a statement about a family of systems, indexed by the temperature. This cannot come out of the gas law and implies an independent axiom.

If we adopt the simplest expression for the hamiltonian,

$$H = \int d^3x (\bar{v}^2/2 + V), \quad V = \mathcal{R}T\rho \log \rho,$$

to serve as “internal energy”, and repeat the analysis of the effect of adiabatically changing the volume by means of an applied pressure, then we shall get

$$pd\mathcal{V} + dH(T, \mathcal{V}) = 0, \quad p = \mathcal{R}TM/\mathcal{V}.$$

In the static case $H = \mathcal{R}TM \log(M/\mathcal{V})$ and

$$dH = \mathcal{R}M \log(M/\mathcal{V})dT - \mathcal{R}TMd\mathcal{V}/\mathcal{V}.$$

The second term compensates for $pd\mathcal{V}$ and so $dT = 0$, the temperature does not change. This is perfectly consistent with the theory as it has been developed so far, but it contradicts experimental results for an ideal gas. Besides, variation of our present, isothermal lagrangian with respect to T does not give a reasonable result, the lagrangian needs to be improved.

II.5. The adiabatic lagrangian

In this section we shall treat the adiabatic gas; that is, the polytropic atmosphere with $\gamma' = \gamma$. But at the end we shall make the case for extending the results to all polytropes.

The two relations $p = \mathcal{R}T\rho$ and $p = \hat{a}\rho^\gamma$ imply the relation $\mathcal{R}T = \hat{a}\rho^{1/n}$ between the two independent variables T and ρ that holds for a set of configurations related by adiabatic transformations. The index n may be fixed for all configurations, while the coefficient \hat{a} parameterizes the family of adiabats (polytropes).

It is possible to derive both relations from a principle of least action, by independent variation of both temperature and density, but a lagrangian functional of Φ, ρ and T can only pertain to a single adiabat. An action principle that describes the whole family of adiabats must involve additional variables, variables that from a restricted point of view of the gas appear as sources of heat. *A lagrangian without sources can be interpreted as applying to a single adiabat of an isolated system.*

The application that this study is aimed at is the earthly or a stellar atmosphere. Those systems are not isolated, but they may nevertheless be treated as being formally

isolated when the energy of radiation is included in the hamiltonian and the effect of radiation is taken into account through the imposition of boundary conditions. It is felt that, if external energy sources are going to be invoked, then it is important that we first establish that there is a need to do so. This is why we continue to regard the lagrangian as applying to an isolated system.

Two kinds of additions can be made to the lagrangian without spoiling the equations of motion that are essential to hydrodynamics.

Adding a term independent of ρ and a term linear in ρ we consider

$$\mathcal{L}[\Phi, \rho, T] = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho \log(\rho/\rho_0) + \rho\mu[T] + f[T]. \quad (2.7)$$

The additions do not spoil the relation $p = \mathcal{R}T\rho$, nor the continuity of the current. Variation with respect to T gives

$$\rho\mu'[T] - \mathcal{R}\rho \log(\rho/\rho_0) = -f'[T] = -(4a/3)T^3. \quad (2.8)$$

We have set $f[T] = (a/3)T^4$, in anticipation of the interpretation of this term as the pressure of the photon gas. If the constant a is the Stefan-Boltzmann constant, $a = 7.64 \times 10^{-15} \text{ ergs}/K^4$, as it will be taken to be, then this term is very small in most circumstances and we must have, on shell, $\mu'[T] \approx \mathcal{R} \log(\rho/\rho_0)$. The following expression will be used

$$\mu[T] = n\mathcal{R}T \log \frac{T}{T_0}. \quad (2.9)$$

Eq.(2.8) takes the form

$$\mathcal{R}\left(n + \log \frac{T^n \rho_0}{T_0^n \rho}\right)\rho + \frac{4a}{3}T^3 = 0, \quad (2.10)$$

and in the important case when $n = 3$

$$\mathcal{R}\left(3 + \log \frac{T^3 \rho_0}{T_0^3 \rho}\right)\frac{\rho}{T^3} + \frac{4a}{3} = 0,$$

which is equivalent to Poisson's law $T^3/\rho = \text{constant}$. This reflects the strong affinity that is found between the polytropic ideal gas with $n = 3$ and radiation. The value $n = 3$ has a cosmological significance as well, it is characteristic of the changes in ρ, p, T induced by uniform expansion (Ritter, Emden, see Chandrasekhar 1939, page 48). For other values of n , Eq.(2.10) is a mild modification of the polytropic equation of change in the presence of radiation.

The equation of motion that is obtained by variation with respect to ρ is

$$\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda + \mu[T] = \mathcal{R}T(1 + \log(\rho/\rho_0)). \quad (2.11)$$

Combined with Eq.(2.10) it reduces, in the static case, to

$$\rho(\phi - \lambda) + (1 + \log k/k_0)\mathcal{R}T\rho = 0, \quad k := \rho/T^n, \quad (2.12)$$

which has the same form as the equation (1.18) studied in Section 1.5.

We have thus found an action that, varied with respect to ρ, Φ and T , reproduces all the equations that define the ideal, polytropic gas with $n = 3$. More generally, for any value of n , it describes a gas that has its effective polytropic index increased from the ‘natural’ adiabatic value so that the neglect of convection is justified. In the limit when radiation becomes negligible it fails to account for the onset of convection, so it should not be used in that case. But with this exception it is a mathematical model that incorporates all of the physical insight that was summarized at the end of Section I.6.

We suggest that using the lagrangian (2.7), with (2.8) and (2.9), is preferable to the usual assumption that $\beta := p_{\text{gas}}/p_{\text{tot}}$ is constant. Additional evidence for the aptness of the adiabatic lagrangian is found in the next subsection.

II.6. Energy, pressure and entropy

The hamiltonian density is, in the static case, with the choice (2.8)-(2.9),

$$h = \rho e + \rho \vec{v}^2/2, \quad \rho e = \phi \rho + \mathcal{R}T\rho \log\left(\frac{\rho}{\rho_0} \frac{T_0^n}{T^n}\right) - \frac{a}{3}T^4. \quad (2.13)$$

With the aid of Eq.(2.10) we obtain for the hamiltonian, on shell, when $\phi = 0$ and $\vec{v} = 0$,

$$H_{\text{tot}} = n\mathcal{R}MT + aT^4\mathcal{V}, \quad (2.14)$$

in full agreement with the familiar expression for the internal energy of an ideal gas with adiabatic index n , augmented by the energy density of the radiation field. *This may be the first time that this expression for the internal energy has been related to the hamiltonian of an action principle.*

The pressure was defined alternatively in terms of the potential, or as the on shell value of the lagrangian. We prefer to define the total pressure by the requirement that

$$p_{\text{tot}}d\mathcal{V} + dH_{\text{tot}} = 0. \quad (2.15)$$

Taking $n = 3$ and $\phi = 0$ we have since $T^3\mathcal{V}$ is constant in this case,

$$dH_{\text{tot}} = 3RMdT + a(T^3\mathcal{V})dT = \left[-\mathcal{R}MT/\mathcal{V} - (a/3)T^4 \right] d\mathcal{V}$$

and thus

$$p_{\text{tot}} = \mathcal{R}MT/\mathcal{V} + \frac{a}{3}T^4. \quad (2.16)$$

This result (2.14-16) is very suggestive. It gives the total pressure as the usual pressure of an ideal gas with polytropic index n , augmented by a term that begs to be interpreted as a pressure due to heat itself, which is natural when heat is interpreted in terms of electromagnetic radiation. Its magnitude is one third of the radiative energy density, as expected for the photon gas.

An energy conservation equation follows in standard fashion from the action principle, namely

$$\frac{d}{dt}(\rho \vec{v}^2/2 + \rho e) + \vec{\nabla} \cdot (\rho \vec{v} \dot{\Phi}) = 0.$$

On the trajectory, $\rho \dot{\Phi} = \rho \vec{v}^2/2 + h + p$ and the standard conservation law results (Castor 2004, page 14), in the case that there is no contribution from heat flow.

The internal energy density, with the aid of (2.10) can be expressed as

$$\rho e = n \mathcal{R} T \rho + \frac{a}{3} T^4.$$

If n is replaced by n' , then this is not in agreement with the energy density for a perfect gas with adiabatic index n :

$$\rho e = n \mathcal{R} T \rho + \frac{a}{3} T^4 + (n' - n) \mathcal{R} T \rho.$$

The discrepancy (the last term) indicates that, when $n' > n$ (the case of convective stability), if a parcel of air is to move down in the atmosphere, to higher densities, then heat has to be supplied at a rate that is proportional to the change in temperature. This is precisely the usual interpretation of polytropic change. Namely, one considers a change dT in temperature that is accompanied by an addition of heat $c \rho dT$. This is a displacement along the polytrope with index

$$\gamma' = \frac{C_P + c}{C_V + c},$$

or $n' = n + c/\mathcal{R}$. Hence

$$\rho e = n \mathcal{R} T \rho + c \rho T.$$

The increase in energy is thus accounted for; it implies that there is an extra supply of energy stored in the gas, distinct from the internal energy and distinct from the contribution $(a/3)T^4$. This tallies very well with the interpretation of the temperature gradient as a greenhouse effect. The extra energy is supplied by the outgoing flux.

II.7. Virial theorem

Both (2.14) and (2.16) are usually derived from considerations outside the proper domain of thermodynamics. We prefer an axiomatic foundation of thermodynamics that is complete in the sense that it does not need other input. As an example let us discuss the use of the virial theorem to make certain predictions concerning stability.

The virial theorem was introduced into the present context by Kelvin. It is based on the scaling properties of the hamiltonian of a system of particles. If $H = K + V$, kinetic energy plus potential energy, then the lagrangian is $K - V$ and the equations of motion imply that, up to a time derivative,

$$\sum_i m_i \dot{q}_i^2 = 2K = - \sum_i q_i \partial_i V.$$

In the case examined by Kelvin the potential is homogeneous of degree -1, so that, in the case of periodic motion, when average is taken over a period, $V = 2K$. According to Chandrasekhar (1938) (pp. 49-51), who also quotes Poincaré, the internal energy is the kinetic energy associated with the microscopic motion of the molecules. It is assumed, usually without discussion, that the presence of gravitational forces do not affect the internal energy, and that the total energy is obtained by simply adding the gravitational potential energy to it. In the present approach there is no place for this argument, the hamiltonian is the energy and there is only one energy.

There is; however, a virial theorem associated with a lagrangian of the type (2.7), that we abbreviate as

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2) - \hat{V}.$$

(The potential \hat{V} includes the gravitational field.) Variation of Φ and of ρ give the equations of motion

$$\dot{\Phi} = \vec{v}^2/2 + (d\hat{V}/d\rho), \quad \dot{\rho} = -\text{div}(\rho\vec{v}),$$

which implies that

$$\int d^3x \frac{d}{dt}(\rho\Phi) = \int d^3x \left(\rho \frac{d\hat{V}}{d\rho} - \rho\vec{v}^2/2 \right).$$

If the system goes through a cycle then the average of this quantity over the cycle is zero,

$$\langle \int dx \rho\vec{v}^2/2 \rangle = \langle \int dx \rho \frac{d\hat{V}}{d\rho} \rangle. \quad (2.17)$$

In the case of (2.10) we obtain, when $n = 3$,

$$\langle \int dx \rho\vec{v}^2/2 \rangle = \langle \int dx \left(\rho(\phi - \lambda) + 4\mathcal{R}T\rho + \frac{4a}{3}T^4 \right) \rangle. \quad (2.18)$$

With Eq.(2.13) this simplifies to

$$\langle \int dx \rho\vec{v}^2/2 \rangle = \langle \int dx \left(\rho(\phi - \lambda) + \mathcal{R}(1 + \log k)\rho T \right) \rangle. \quad (2.19)$$

This result, like classical virial theorems, applies exclusively to the case of periodic motion.

In the special case $\vec{v} = 0$ Eq.(2.19) is a direct consequence of the equations of motion. Such relations, that do not depend on the periodicity of the motion, are not true virial theorems.

II.8. The centrifuge and the atmosphere

Kelvin justified the polytropic model of the atmosphere in terms of radiation and convection. Eddington discounted the role of convection and relied on a concept of radiative equilibrium. To find out what happens in the case of complete insulation we study the analogous situation in a centerfuge.

Consider an ideal gas. By a series of experiments in which gravity does not play a role, involving reversible changes in temperature and pressure, it is found that, at equilibrium, the laws $p/\rho = \mathcal{R}T$ and $\rho = kT^n$ are satisfied, constants k, n fixed. When supplemented by the laws of hydrodynamics, they are found to hold, or at least they are strongly believed to hold, in configurations involving flow, over a limited time span, in the absence of external forces. In addition it is said that, at equilibrium, the temperature must be uniform. Keeping an open mind, let us refer to this last statement as “the axiom”. We are talking about a fixed quantity of gas contained in a vessel, the walls of which present no friction and pass no heat.

Let the walls of the vessel be two vertical, concentric cylinders, and construct a stationary solution of the equations of motion. And why not? We have experimental confirmation of the equations of motion, we applied them to the theory of sound with a degree of confidence that is so high that the prediction of rapid variations in temperature may never have been subjected to verification (?). In terms of cylindrical coordinates, take $v_z = v_r = 0, v_\theta = \omega$, constant. The continuity equation is satisfied with ρ any function of r alone. Then neither T nor p is constant, for the hydrodynamical equations demand that

$$r\omega^2 = cT', \quad c = (n+1)\mathcal{R} \approx 10^7 \text{ cm/sec}^2 K \quad (\text{for air}).$$

At first sight, this seems to violate the axiom, but perhaps not, for this is not a static configuration. To save the axiom let us suppose that, by conduction, convection or radiation, the temperature will tend towards uniformity. Perhaps after a suitably long time has passed, T has become constant, in violation of the equations of motion. Let us remember that no heat or any other influence is supposed to go by the walls; then surely energy and angular momentum must both be preserved during the time that the temperature is leveling out. It also seems reasonable to assume that the final configuration is (macroscopically) stationary and uniform, since the existence of fluctuations would imply that the entropy had not reached its maximum. But a stationary state with non zero density gradient and uniform temperature would seem to contradict the assumptions that we made about the gas, which makes the existence of such a state somewhat problematic.

If we also accept the equivalence principle, then from the point of view of a local observer at rest in the flow there is a centrifugal force field, a density gradient and, by the laws of Poisson, a temperature gradient. The equivalence principle only applies to conditions at one point, and one can question whether the gradient of the temperature or of the density is sufficiently local to be covered by the principle. The entire theory of relativistic thermodynamics has been founded on the belief that it is (Tolman 1934).

If we do accept the equivalence principle (without necessarily embracing the tenets of traditional relativistic thermodynamics), then we shall be lead to expect that a vertical column of an ideal gas, in mechanical equilibrium under the influence of terrestrial gravity, and perfectly isolated, will have a pressure and temperature gradient exactly of the form

predicted by Lane. This seems to contradict what we think is the prevailing opinion of atmospheric scientists, that the temperature gradient owes its existence to the heating associated with solar radiation.

Further measurements in the atmosphere are unlikely to throw light on this, since isolation is out of the question. Experiments with a centrifuge may be more realistic. The temperature lapse rate is $r\omega^2 \times 10^{-7} K/cm$. If the acceleration is $1000 g$ at the outer wall, then the lapse rate will be $.1 K/cm$. The temperature difference between the inner and outer walls will thus be $1 K$ if the distance is $10 cm$. In a practical experiment one does not have the gas flow between concentric, stationary cylinders. Instead a tube filled with the gas is oriented radially on a turntable. Friction against the walls is thus eliminated and heat loss is much easier to control.

On purely theoretical grounds we have come to doubt that complete equilibrium implies a uniform temperature in all cases. In fact, Tolman (1934, page 314) shows that, according to General Relativity, the temperature of an isolated photon gas in a gravitational field is not quite uniform. The predicted magnitude of this effect is very small, but it shows that there are circumstances in which statistical mechanics is not the absolute truth.

The strongest argument that we have found, against the indefinite persistence of a temperature gradient in an isolated system, is this. Imagine a large heat bath located in the region $z > 0$. A vertical tube, filled with an ideal gas, has its upper end in thermal contact with the bath, otherwise it is isolated. Assume that, at equilibrium, the lower part of the tube has a temperature that is higher than that of the bath, then we can run an engine by taking out a small amount of heat from the bottom of the tube and returning it to the bath. In order for equilibrium to be restored, the heat thus taken out of the tube will have to be restored from the bath, which implies a spontaneous transfer of heat in the direction opposed to the gradient of the temperature, in violation of one of the statements of the second law (Clausius 1887). According to Graeff (2008), this is exactly what he observes!

If our analysis of the centrifuge is incorrect, what is the mistake? The conventional view is that, if a temperature gradient exists at one time, then heat flow will tend to dissipate it. Dissipation is determined by Fourier's heat equation; in its general form

$$\frac{D}{Dt}T = \partial_i C^{ij} \partial_j T. \quad (2.20)$$

Our error consists of the fact that we are using a lagrangian that contains neither space derivatives nor time derivatives of the temperature. We are neglecting phenomena that have a long time scale, such as conduction and dissipation. This can be justified if we limit our attention to stationary configurations, when $\dot{T} = 0$, if the temperature distribution makes the right hand side of the equation equal to zero. But a more complete understanding of the phenomena requires that we consider nonstationary situations as well, and for that we shall have to include additional terms in the action, terms that become significant when radiation is not.

It is not just the paradox of the centrifuge that forces us in this direction. It is important that the theory should account for a situation in which the radiation is subject to temporal variations, as is certainly the case for the earthly atmosphere.

III. Sources

III.1. Generic source

We have proposed to extremize the lagrangian with respect to all three fields, Φ , ρ and T . We have found an expression for the adiabatic lagrangian density, the Euler-Lagrange equations of which pass the two tests: 1. When the effect of radiation pressure is neglected they give precisely the equations that govern the polytropic atmosphere, for all n . 2. In the case that $n = 3$ and with the radiation term included, they reduce to the equations of radiative equilibrium.

The adiabatic lagrangian describes a single adiabat. To remove this limitation in a formal way, let us add another term to the lagrangian density,

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - gz + \lambda) - \mathcal{R}T\rho \log \frac{k}{k_0} + f[T] + \rho TS, \quad (3.1)$$

where S is an external source. The factor ρ in the source term is natural and the factor T is chosen to make S play the role of a local adiabatic parameter. We have introduced the variable k and the parameter k_0 by

$$\rho = kT^n, \quad \rho_0 = k_0 T_0^n, \quad (k_0 \rightarrow 1);$$

Then $k^{-1/n}$ is Emden's "polytropic temperature". It will be recalled that k_0 parameterizes a family of adiabats; in fact, for an isothermal expansion, the variation of $-\mathcal{R} \log k$ is precisely the change in specific entropy. The introduction of the source S turns $-\mathcal{R} \log k_0$ into a field with the interpretation of entropy. We no longer need the parameter and so, following Lane, we shall use units of density such that $k_0 = 1$.

The internal specific entropy is $\mathcal{R} \log(T^n/\rho)$ and the total specific entropy is

$$S_{tot} = \mathcal{R} \log \frac{T^n}{\rho} + S.$$

With this convention

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) + \rho T S_{tot} + \frac{a}{3} T^4, \quad (3.1)$$

where ϕ is the gravitational potential. Variation with respect to T leads to

$$\rho \frac{\partial}{\partial T} (T S_{tot}) + \frac{4a}{3} T^3 = 0. \quad (3.2)$$

As an equation for S_{tot} it has the general solution

$$S_{tot} = -\frac{a}{3\rho} T^3 - \frac{1}{\rho T} V[\rho]. \quad (3.3)$$

Taking this as the definition of the potential (V is the value of $\rho T S_{tot} + \frac{a}{3} T^4$ at the extremum with respect to variation of T) we have

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - V[\rho]. \quad (3.4)$$

The gradient of the equation obtained by variation of ρ is

$$-\rho \frac{D}{Dt} \vec{v} - \rho \text{grad } \phi = \text{grad } p, \quad (3.5)$$

with

$$p = \rho V' - V = (1 - \rho \frac{d}{d\rho})(\rho T S_{tot} + \frac{a}{3} T^4) = -T \rho^2 \frac{\partial S_{tot}}{\partial \rho} + \frac{a}{3} T^4. \quad (3.6)$$

The last equation is justified by the fact that the partial derivative of $\rho T S_{tot} + (a/3) T^4$ with respect to T vanishes, Eq.(3.2).

We shall verify some important relations of thermodynamics, and for this we must take T and ρ to be constant, with $M = \rho \mathcal{V}$, and $\phi = 0$. In this case

$$p = MT \frac{\partial S_{tot}}{\partial \mathcal{V}} + \frac{a}{3} T^4 = \mathcal{R} MT / \mathcal{V} + \frac{a}{3} T^4 + MT \frac{\partial S}{\partial \mathcal{V}}. \quad (3.7)$$

The hamiltonian density is, in the static case, in the absence of gravity,

$$h = -\rho T S_{tot} - \frac{a}{3} T^4, \quad \text{implying that } U = -MT S_{tot} - \frac{a}{3} T^4 \mathcal{V}. \quad (3.8)$$

Variation of h with respect to T gives zero on shell, so this is the same as

$$u = (1 - T \partial_T) h = \rho T^2 \frac{\partial S_{tot}}{\partial T} + a T^4 = n \mathcal{R} \rho T + a T^4 + \rho T^2 \frac{\partial S}{\partial T}.$$

Thus

$$U = MT^2 \frac{\partial S_{tot}}{\partial T} + a T^4 \mathcal{V} = n \mathcal{R} MT + a T^4 \mathcal{V} + MT \frac{\partial S}{\partial T}. \quad (3.9)$$

Using (3.7) and (3.8) one verifies that

$$\frac{\partial U}{\partial \mathcal{V}} = (T \partial_T - 1) p, \quad (3.10)$$

an important consequence of the existence of entropy in general. See Finkelstein (1969) page 26. Also,

$$dU = MT \frac{\partial S_{tot}}{\partial T} dT + T d\left(MT \frac{\partial S_{tot}}{\partial T}\right) + 4a T^3 \mathcal{V} dT + a T^4 d\mathcal{V},$$

$$pd\mathcal{V} = MT \frac{\partial S_{tot}}{\partial \mathcal{V}} d\mathcal{V} + \frac{a}{3} T^4 d\mathcal{V},$$

and the sum is $dU + pd\mathcal{V} = dQ = T(\partial p / \partial T) = 0$, the last on shell.

We used the last expressions in (3.7) and (3.9) because they are familiar, but if we return to (3.8) and the first expression for p in (3.7) we see immediately that $dU + pd\mathcal{V} = 0$.

If instead we consider a change that involves outside forces acting via the source, then $dU + pd\mathcal{V} = MT \delta S$, which confirms the interpretation of S as a contribution to the specific entropy.

The calculations that have been presented in this subsection are offered as proof that the variational approach that is being advocated is fully compatible with classical thermodynamics. This gives us faith in the basic framework and courage to proceed on a more speculative course.

III.2. Electromagnetic fields

We write the Maxwell lagrangian as follows,

$$\mathcal{L}_{\text{rad}} = \frac{1}{2\epsilon} \vec{D}^2 - \frac{\mu}{2} \vec{H}^2 + \vec{D} \cdot (\vec{\partial} A_0 - \dot{\vec{A}}) - \vec{H} \cdot \vec{\partial} \wedge \vec{A} + JA, \quad (3.13)$$

and add it to the ideal gas lagrangian

$$\mathcal{L}_{\text{gas}} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho \log k + \frac{a}{3}T^4, \quad (3.14)$$

Since the susceptibility of an ideal gas is small, the dielectric constant may be expressed by

$$\epsilon = 1 + \kappa[\rho, T], \quad \text{or} \quad \frac{1}{\epsilon} = 1 - \kappa[\rho, T]. \quad (3.15)$$

Paramagnetic effects will be ignored at present. An interaction between the two systems occurs through the dependence of the susceptibility on ρ . The source S has become $-(\vec{D}^2/2\rho)(\kappa/T)$. If this quantity has a constant value then it produces a shift in the value of the parameter k .

Two interpretations are possible. The electromagnetic field may represent an external field, produced mainly by the source J , and affecting the gas by way of the coupling implied by the dependence of the dielectric constant on ρ . Alternatively, $J = 0$ and the field is produced by microscopic fluctuations, quantum vacuum fluctuations as well as effects of the intrinsic dipoles of the molecules of the gas. In this latter case the main effect of radiation is represented by the radiation term $aT^4/3$. Our difficulty is that neither interpretation is complete, and that we do not have a sufficient grasp of the general case when either interpretation is only half right. The following should therefore be regarded as tentative.

Variation of the total action, with lagrangian $\mathcal{L}_{\text{rad}} + \mathcal{L}_{\text{gas}}$, with respect to $\vec{A}, \vec{D}, \vec{H}$ and T gives

$$\dot{\vec{D}} = \vec{\partial} \wedge \vec{H}, \quad (3.14)$$

$$\dot{\vec{A}} = \vec{D}/\epsilon, \quad (3.15)$$

$$\mu \vec{H} = -\vec{\partial} \wedge \vec{A}, \quad (3.16)$$

and

$$\mathcal{R}(n - \log k)\rho - \frac{\vec{D}^2}{2} \frac{\partial \kappa}{\partial T} + \frac{4a}{3}T^3 = 0. \quad (3.17)$$

Taking into account the first 3 equations we find for the static hamiltonian

$$H = \int d^3x \left(\phi\rho + \mathcal{R}\rho T \log k + \frac{\vec{D}^2}{2} + \frac{\mu \vec{H}^2}{2} - \frac{\vec{D}^2}{2} \frac{\kappa}{T} - \frac{a}{3}T^4 \right).$$

With the help of (3.17) it becomes

$$H = \int d^3x \left(\phi \rho + n \mathcal{R} \rho T + \frac{\vec{D}^2}{2} + \frac{\mu \vec{H}^2}{2} + a T^4 \right) - \int d^3x T \frac{\vec{D}^2}{2} \frac{\partial(T\kappa)}{\partial T}. \quad (3.18)$$

The last term, from the point of view of the thermodynamical interpretation of electrostatics, is recognized as the entropy (Panofsky and Phillips 1955). On a suitable choice of the functional κ it merges into the internal energy. For example, if $\kappa = \rho T$ it takes the form $\rho T S$ with $S = \vec{D}^2$.

III.3. Discussion 1. Using T as a dynamical variable

The idea of extremizing thermodynamical potentials with respect to the temperature is far from new, but in the context of the action principle it is likely to raise questions.

Radiation Hydrodynamics (Castor 2004) is defined by the following equations (without specialization to laminar flow): the continuity equation, the hydrodynamical equation, the equation of motion, and an ‘energy equation’. This last equation takes the place of the equation that results from variation of the temperature. To see this we have only to review the canonical conservation of energy in lagrangian/hamiltonian form.

With \mathcal{L} as in (3.4),

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - V[\rho, T],$$

with no derivatives in the functional $V[\rho, T]$. We have

$$\frac{d\mathcal{L}}{dt} = \dot{\rho} \frac{\partial \mathcal{L}}{\partial \rho} + \dot{\Phi} \frac{\partial \mathcal{L}}{\partial \Phi} + \text{grad } \Phi \cdot \frac{\partial \mathcal{L}}{\partial \text{grad } \Phi} + \dot{T} \frac{\partial \mathcal{L}}{\partial T}.$$

On shell, the first term on the right is zero. The equation can be rearranged to read

$$\frac{Dh}{Dt} + \text{div}(p \vec{v}) = -\dot{T} \frac{\partial \mathcal{L}}{\partial T}.$$

The ‘energy equation’ is the vanishing of the left side; it is thus equivalent to setting $\partial \mathcal{L} / \partial T = 0$. There is no evidence of a contribution to energy transport by heat flow, but that is not a problem if we accept Emden’s postulate of a constant heat flow.

The effect of heat transfer by radiation and by conduction has been left out, for simplicity and since it does not immediately affect the polytropic atmosphere. But it means that our treatment is incomplete. Most important, it prevents us from dealing with the isothermal atmosphere that is, after all, an important special case. Consequently, it is felt that a general understanding of the effect of radiation is lacking.

Let us examine the total lagrangian,

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\text{rad}} + \mathcal{L}_{\text{gas}} = & \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R} T \rho \log \frac{\rho}{T^3} \\ & + \frac{\vec{D}^2}{2\epsilon} + \frac{\mu}{2} \vec{H}^2 + \vec{D} \cdot (\vec{\partial} A_0 - \dot{\vec{A}}) - \vec{H} \cdot \vec{\partial} \wedge \vec{A} + J A + \frac{a}{3} T^4. \end{aligned} \quad (3.19)$$

So long as ϵ, μ and J are independent of ρ, T and \vec{v} , the variational equations of motion that are obtained by variation of $\vec{v}, \rho, \vec{A}, \vec{H}$ and \vec{D} are all conventional, at least when $n = 3$ (for all n if radiation is negligible). It would be possible to be content with that and fix T by fiat, as is usual; in the case of the ideal gas without radiation the result is the same. But if ϵ depends on ρ and on T , which is actually the case, then we get into a situation that provides the strongest justification yet for preferring an action principle formulation with T as a dynamical variable. The equations of motion include a contribution from the variation of ϵ with respect to ρ , so that one of the basic hydrodynamical equations is modified. Thus it is clear that the extension of the theory, to include the effect of radiation, is not just a matter of including additional equations for the new degrees of freedom. The presence of the term $\vec{D}^2/2\epsilon[\rho, T]$ certainly introduces the density ρ into Maxwell's equations; that it introduces \vec{D} into the hydrodynamical equations is clear as well. *The over all consistency of the total system of equations can be ensured by heeding Onsager's principle of balance, but the action principle makes it automatic.*

Variation of the action with respect to T offers additional advantages. The usual procedure, that amounts to fixing $\rho = kT^n$, k and n constant, gives the same result when radiation is a relatively unimportant companion to the ideal gas, but in the other limiting case, when the density is very dilute and the gas becomes an insignificant addition to the photon gas, it is no longer tenable. We need an interpolation between the two extreme cases and this is provided naturally by the postulate that the action is stationary with respect to variations of the temperature field.

In the absence of the ideal gas we have another interesting system, the pure photon gas. The analogy between the photon gas and the ideal gas is often stressed; there is an analogue of the polytropic relation that fixes the temperature in terms of ρ ; the pressure of the photon field is $(a/3)T^4$. Our lagrangian already contains this pressure; we should like to discover a closer connection between it and the electromagnetic field. In the limit when the density of the ideal gas is zero, Eq.(3.17) becomes

$$-\frac{\vec{D}^2}{2} \frac{\partial \kappa}{\partial T} + \frac{4a}{3} T^3 = 0.$$

In the absence of the gas it is reasonable to impose Lorentz invariance, so we include magnetic effects by completing the last to

$$-\frac{F^2}{2} \frac{\partial \kappa}{\partial T} + \frac{4a}{3} T^3 = 0.$$

If we suppose that $\kappa[\rho, T]$, in the limit $\rho = 0$, takes the form αT^2 , then

$$\alpha F^2 = \frac{4a}{3} T^2.$$

The radiation from a gas of Hertzian dipoles can be shown, with the help of the Stefan-Boltzman law and Wien's displacement law, to satisfy a relation of precisely this form. Whether the same relation holds in vacuum is uncertain, but it is suggested by an analysis

of the effective Born-Infeld lagrangian calculated on the basis of the scattering of light by light (Euler 1936, Karplus and Neuman 1950). See also McKenna and Platzman (1962).

III.4. Discussion 2. The temperature gradient of the atmosphere

We return to the question of the heating of the atmosphere by solar radiation. The susceptibility of air may be approximated by $\kappa = \alpha\rho/T$ (α constant); in which case the lagrangian includes the following

$$-\mathcal{R}T\rho\log\rho + n\mathcal{R}\rho T\log T + \alpha\rho T^{-1}\vec{D}^2/2. \quad (3.20)$$

Up to a factor ρT , the first term is the entropy of the isothermal gas. The first two terms together give the entropy of the polytropic gas. The entropy introduced by radiation is $\alpha\vec{D}^2(\partial(\kappa T)/\partial T)$, namely zero under the assumptions made, which would suggest that the greater effect of radiation on the specific entropy of air comes about through the influence of the term $aT^4/3$ in the lagrangian. The second term was included in our theory in order that variation of the action give both of the Poisson relations, the ideal gas law and $\rho \propto T^n$. Let us return to the problem of determining the origin of the temperature gradient, the physical reason for the fact that the atmosphere is nearly adiabatic. In the model, adiabacity comes with the inclusion of the term $n\mathcal{R}\rho T\log T$ in the action. If this term could be linked with the last term in (3.20); that is, if it could be shown that $\alpha\vec{D}^2/2 \approx n\mathcal{R}T^2\log T$, then we could conclude that the temperature profile is created by the radiation. But this seems unlikely, as it amounts to fine tuning. If we suppose that the term in question is responsible for the bulk of the effect of solar radiation on the atmosphere of the earth, then we must admit that it undergoes important diurnal and seasonal variation. In fact, the polytropic index of the troposphere is usually quoted as a constant, without any indication that important variations have been observed over a period of time.

It could be argued that the relatively stable polytropic index of the earth is evidence that the atmosphere, if isolated, would continue to manifest a temperature gradient and that an ideal gas, isolated in a gravitational field, may not tend to an equilibrium state with uniform temperature, but the explanation of the temperature gradient in terms of convection is more attractive by far. See the last paragraph of Section I.6. Additional terms (containing in particular derivatives of the temperature) are needed in the lagrangian, although they are insignificant in applications to the polytropic atmosphere under conditions of strong radiation fields.

IV. Stability of atmospheres

IV.1. The isothermal column

We consider the space that is tangent to a static solution with density ρ_0 . Setting $\rho = \rho_0 + \delta\rho$ we have the following equations for the perturbation $\delta\rho$,

$$-\dot{v} = \mathcal{R}T(\delta\rho/\rho_0)', \quad \delta\dot{\rho} = -(\rho_0 v)',$$

where the prime denotes differentiation with respect to z . Thus

$$\delta\ddot{\rho} = \mathcal{R}T(\rho_0\alpha')', \quad \alpha := \delta\rho/\rho_0. \quad (4.1)$$

For a harmonic mode with frequency ω , $-\omega^2\delta\rho = \mathcal{R}T(\rho_0\alpha')'$, and

$$-\frac{\omega^2}{\mathcal{R}T} \int \alpha^2 \rho_0 dz = \int \alpha(\rho_0\alpha')' dz = \delta\rho \alpha' \Big|_0^\infty - \int \rho_0 \alpha'^2 dz.$$

The configuration is stable if this implies that $\omega^2 > 0$, which will be the case if the boundary term vanishes. To justify any choice of boundary conditions we have only the conservation of mass, $\int \delta\rho dz = 0$. This ensures that $\delta\rho$ fall off at infinity and we are left with $-\delta\rho(0)\alpha'(0)$.

We shall show that $\alpha'(0) = 0$. Eq.(4.1) tells us that

$$-\frac{\omega^2}{\mathcal{R}T} \alpha = (\rho_0'/\rho_0)\alpha' + \alpha'' = -\frac{g}{\mathcal{R}T} \alpha' + \alpha''.$$

This is a linear differential equation with constant coefficients, with general solution

$$\alpha = Ae^{k_+z} + Be^{k_-z}, \quad k_\pm = \frac{g}{2\mathcal{R}T} \pm \sqrt{\left(\frac{g}{2\mathcal{R}T}\right)^2 - \frac{\omega^2}{\mathcal{R}T}}.$$

Since, up to an irrelevant constant factor, $\rho_0 = \exp(-gz/\mathcal{R}T)$,

$$\delta\rho = \rho_0\alpha = Ae^{a_+z} + Be^{a_-z}, \quad a_\pm = -\frac{g}{2\mathcal{R}T} \pm \sqrt{\left(\frac{g}{2\mathcal{R}T}\right)^2 - \frac{\omega^2}{\mathcal{R}T}}.$$

These functions are integrable only if $0 < \omega^2 < g^2/4\mathcal{R}T$, and in that case

$$\delta M = \int \delta\rho dz = -\frac{A}{a_+} - \frac{B}{a_-} = \frac{A}{k_-} + \frac{B}{k_+},$$

the vanishing of which requires that $\alpha'(0) = 0$. When $\omega^2 > g^2/4\mathcal{R}T$ we have instead to do with a contour integral, and reach the same conclusion. Therefore, not only is the condition $\omega^2 > 0$ verified; it is also confirmed that the boundary condition $\alpha'(0) = 0$ is the only one possible. We have seen that this choice of boundary conditions is the one that ensures the conservation of mass.

IV.2. The polytropic column

Let us leave the parameter $k = \rho/T^n$ free and fix the value of n . This conforms to the usual approach when the temperature is fixed by edict, but it is consistent with our formulation if $n = 3$ only. We study the stability to vertical perturbations.

The static solution is

$$cT = \lambda - gz, \quad c := \mathcal{R}(1 + \log k).$$

A first order perturbation satisfies

$$\dot{\Phi} + \delta\lambda = c\delta T, \quad \text{thus} \quad \dot{v} = -c\delta T', \quad (IV.1)$$

and

$$\dot{\rho} = -(v\rho)', \quad \ddot{\rho} = -(\dot{v}\rho)', \quad \text{and} \quad nT^{n-1}dT = c(T^n\delta T')'. \quad (IV.2)$$

Let $x = \lambda/g - z$, $0 < x < \lambda/g$ and let $f' = df/dx$ from now on. Solutions of the type $\delta T = \exp(i\omega t)f(x)$ satisfy the equation

$$(x^n\delta T')' + \frac{\nu^2}{x}(x^n\delta T) = 0, \quad \nu^2 = n\omega^2/g. \quad (IV.3)$$

The solution that is regular at the origin of x (the top of the atmosphere) is

$$\delta T = {}_0F_2(n, -\nu^2 x)e^{i\omega t}.$$

The generalized hypergeometric function is positive for positive argument and it oscillates around zero for negative argument.

Boundary conditions. If we fix $\delta T = 0$ at the bottom of the column we can prove stability as follows. For a harmonic perturbation,

$$\nu^2 \int x^{n-1}(\delta T)^2 dx = - \int (x^n\delta T')'\delta T dx = \int x^n(\delta T')^2 dx > 0,$$

which shows that ν^2 is positive and that the solutions are oscillatory in time. But there is no justification for this choice of boundary condition.

It is not unusual to fix the upper boundary, and to require that the perturbation vanish there. If ν^2 is positive the argument of the hypergeometric function is negative. The function oscillates around zero and for a discrete set of values of the frequency it vanishes at the upper end. If ν^2 is negative then the hypergeometric function is positive, and the boundary condition cannot be met.

However, there seems to be no better reason to fix the upper boundary. The natural boundary condition is that the mass must be preserved, thus

$$\delta M = \int \delta\rho dx = \int T^{n-1}\delta T dx = 0.$$

This may happen for a discrete set of positive values of ν^2 . For negative values of ν^2 the integrand is definite so that it can not happen. The calculation is valid only in the case $n = 3$; this atmosphere is stable. For other values of n the calculation is more difficult.

The problem can be converted to a standard boundary value problem by rescaling of the coordinate.

The mass is

$$M = Ak\left(\frac{g}{a}\right)^3 \int dx x^3 = \frac{Ak}{4}\left(\frac{\lambda}{a}\right)^3 \frac{\lambda}{g}. \quad (IV.4)$$

The proper definition of gravitational energy is ambiguous, but (II.21) suggests that it is

$$E_g = Ag \int \rho(gz - \lambda) dz = -Ak g \left(\frac{g}{a}\right)^3 \int_0^{\lambda/g} dx x^4 = -\frac{Ak\lambda}{5} \left(\frac{\lambda}{a}\right)^3 \frac{\lambda}{g}$$

The last expression, and those that follow, refer to the static solutions. The thermodynamic part of the hamiltonian is

$$\begin{aligned} H - E_g &= A \int r^2 dr (\mathcal{R}T\rho \log k + \frac{a}{3}T^4) = \frac{Ak c}{4} \int_R^{\lambda/g} dx T^4 \\ &= \frac{Ak\lambda}{20} \left(\frac{\lambda}{a}\right)^3 \frac{\lambda}{g} = \frac{-1}{4} E_g. \end{aligned} \quad (IV.5)$$

The integrand on the right hand side of (II.21) is thus $E_g + 4(H - E_g) = 0$, as it must be.

IV.3. The polytropic gas sphere. The hamiltonian

Here we study the self gravitating polytropic gas. A correction is needed in the expression for the lagrangian, and we need to take care with respect to the definition of the gravitational potential.

First of all, it would not be difficult to argue that the correct expression for the gravitational energy is

$$E_g = -\frac{G}{2} \int d^3x d^3x' \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad (IV.6)$$

and that, consequently, the term $-gz$ in the lagrangian has to be replaced by $-E_g$. Nothing else is needed, but to make an important point it will be useful to introduce a potential, a functional $\phi[\rho]$ defined by

$$\phi[\rho](\vec{x}) = \phi(\vec{x}) = \int d^3x' \frac{G\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} + \phi[0]. \quad (IV.7)$$

The last term, $\phi[0] := \phi[\rho]|_{\rho=0}$ is of course an arbitrary constant field. The value chosen for this constant is irrelevant, but it must be kept in mind that it is chosen once and for all and that it is independent of ρ . The sign is opposite to that used by Eddington; it is chosen so that the gravitational force is $-\text{grad } \phi$.

For any spherically symmetric distribution define

$$\mathcal{M}(r) = 4\pi \int_0^r r'^2 \rho(r'); \quad (IV.8)$$

then

$$\phi(\vec{x}) \rightarrow \phi(r) = \int_0^r \frac{G\mathcal{M}}{r'^2} dr' + \phi(0), \quad (IV.9)$$

$$r^2 \phi' = G\mathcal{M}, \quad \mathcal{M}' = 4\pi r^2 \rho. \quad (IV.10)$$

Since first offered by Perry (1899) it has become customary to present a criterion for the stability of static configurations, based on an evaluation of the energy. The better to understand it we replace the definition (4.6) of the gravitational energy by

$$E_g = \frac{1}{2} \int \rho \phi d^3x = \frac{1}{2} \int \mathcal{M}' \phi dr, \quad (IV.11)$$

acknowledging that, since ϕ is defined up to an additive constant, the same is true of the energy.

As we have seen in Section II.7, it is reasonable to identify the total energy (including the gravitational energy and the internal energy) with the hamiltonian. The lagrangian is

$$L = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi/2 + \lambda) - \mathcal{R}T\rho \log k - \frac{a}{3}T^4. \quad (IV.12)$$

The factors of 1/2 in the term $\rho\phi/2$ and in (IV.11) arise from the fact that $\rho\phi$ is a homogeneous functional of ρ of order 2, $\delta(\rho\phi/2) = \phi$. Variation of the action with respect to T gives the relation

$$\mathcal{R}T(n - \log k) = \frac{4a}{3} \frac{T^4}{\rho}. \quad (IV.13)$$

The hamiltonian density is

$$h = \rho\vec{v}^2/2 + \frac{1}{2}\rho\phi + \mathcal{R}T\rho \log k + \frac{a}{3}T^4,$$

or, in view of (IV.13),

$$h = \frac{1}{2}\vec{v}^2 + \frac{1}{2}\rho\phi + \frac{3c}{4}T\rho, \quad c = \mathcal{R}\left(\frac{n}{3} + \log k\right),$$

whence the hamiltonian (= total energy)

$$H = \int d^3x h = E_g + \int d^3x \left(\frac{1}{2}\vec{v}^2 + \frac{3c}{4}T\rho\right). \quad (IV.14)$$

In the static case we have the equation of motion

$$\lambda = \phi + cT. \quad (IV.15)$$

Since the surface of the star is at the point where $T = 0$, it follows that

$$\lambda = \phi(R) = \phi(0) + cT(0). \quad (IV.16)$$

Applying $\int d^3x \rho$ to (IV.15) we obtain

$$M\lambda = 2E_g + \int cT\rho d^3x.$$

Hence (4.14) reduces, in the static case, to

$$H = E_g + \frac{3}{4}(M\phi(R) - 2E_g) = -E_g/2 + \frac{3}{4}M\phi(R). \quad (IV.17)$$

Note that $H = E_g + U$ and that this is what Eddington and others call the total energy.

It remains to calculate E_g , and here we follow Eddington. To begin, using only the definitions (4.8)- (4.11),

$$E_g = 4\pi \int_0^R \frac{1}{2} \rho \phi r^2 dr = \frac{1}{2} \int \mathcal{M}' \phi dr = \frac{1}{2} M \phi(R) - \frac{1}{2} \int \mathcal{M} \phi' dr, \quad (4.18)$$

and

$$-\frac{1}{2} \int \mathcal{M} \phi' dx = -\frac{1}{2} \int \frac{GM^2}{r^2} dr = \frac{1}{2} \int G\mathcal{M}^2 \left(\frac{1}{r}\right)' dr = \frac{1}{2} \frac{GM^2}{R} - \int \frac{G}{r} \mathcal{M} d\mathcal{M}. \quad (4.19)$$

Next, the polytropic relation can be used to show that

$$\frac{3}{4} \int \mathcal{M} \phi' dr = \int \frac{G}{r} \mathcal{M} d\mathcal{M}.$$

Hence $\int \mathcal{M} \phi' dr = 2GM^2/R$ and finally

$$E_g = \frac{1}{2} M \phi(R) - \frac{GM^2}{R}, \quad H = \frac{1}{2} M \phi(R) + \frac{1}{2} \frac{GM^2}{R}. \quad (4.20)$$

So far, the only difference between our calculation and those of Eddington and others is the fact that we have left open the zero point of the potential. Eddington's field is $\phi(\text{Eddington}) = -\phi - GM/R$ and his boundary condition $\phi(\text{Eddington}) = 0$ at the surface, amounts to

$$\phi(R) = -\frac{GM}{R}. \quad (\text{Eddington's choice})$$

According to (4.16) this is the same as $\phi(0) = -(MG/\mathcal{R}) - cT(0)$. But, as we have emphasized already, $\phi(0) = \phi[0]$ is a constant functional, independent of the dynamical variables. Therefore, Eddington's choice is not only *ad hoc* but, in the context of the action principle, wrong! In fact, we know of no physical theory in which the manifold of physical states is restricted to a single energy surface in phase space.

The total energy provided by the action principle is given by (IV.20) and (IV.16),

$$H[\rho, T] = \frac{1}{2}M \left(\frac{GM}{R} + cT(0) + \phi[0] \right).$$

It depends on the value M chosen for the constant of the motion, the initial value $T(0)$, and the choice of an inessential zero point for the gravitational potential.

We feel justified to conclude that the insistence on an action principle is much more than an aesthetic preference; it is an essential aid to avoid fortuitous conclusions.

We have chosen to investigate the case $n = 3$, since Eddington's calculations are valid in that case only. As was shown, they lead to no conclusion even in that case. The statement that the static configurations are stable for $n < 3$ and unstable for $n \geq 3$ may be correct, but to say that it was proved by the argument first advanced by Perry, or by the same calculation repeated in many of our modern textbooks, is an exaggeration.

IV.4. The polytropic gas sphere. Stability

We use the lagrangian

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi/2 + \lambda) - \mathcal{R}T\rho \log k + \frac{a}{3}T^4, \quad k := \rho/T^n. \quad (IV.21)$$

Variation with respect to T gives

$$\mathcal{R}(\log k - n) = \frac{4a}{3}T^3/\rho. \quad (IV.22)$$

With $n = 3$ this makes k a constant, and $\log k = 3$ when radiation is neglected. In the remainder of this section, we set, for all values of n ,

$$\rho = kT^n, \quad k \text{ constant.}$$

This is the usual polytropic relation used by Eddington and others, but it is consistent with (4.21) only when $n = 3$. The remaining dynamical equations are

$$-\frac{Dv}{Dt} = \phi' + cT', \quad \dot{\rho} + r^{-2}(r^2\rho v)' = 0,$$

$$4\pi G\rho = r^{-2}(r^2\phi')', \quad \rho = kT^n.$$

• The static solution. Eliminate ϕ by $\phi' = -cT'$ and change variables, setting $r = x/\alpha$, α constant, Poisson's equation becomes

$$\frac{4\pi Gk}{c\alpha^2}x^2T^n + (x^2T')' = 0,$$

where the prime now stands for differentiation with respect to x . Set $f(x) = T(x)/T(0)$ and $\alpha = \sqrt{4\pi G/cT(0)}$ so that finally

$$x^2 f^n + (x^2 f')' = 0, \quad f(0) = 1, \quad f'(0) = 0.$$

The solution decreases monotonously to zero at $x = X$, this point taken to be the surface of the star. At the outer limit $f(x) \propto X/x - 1 + o(X - x)^n$. The integration is done easily and accurately by Mathematica, especially so for integer values of n . The radii are, for $n = 2 : X = 4.355$, $n = 3 : X = 6.89685636197$, $n = 4 : X = 14.9715$.

- For the fluctuations we assume harmonic time dependence, then the equations are

$$-\omega^2 r^2 \delta\rho = (r^2 \rho (\delta\phi' + c\delta T'))', \quad \delta\rho = nkT^{n-1} \delta T, \quad (IV.23)$$

$$4\pi G r^2 \delta\rho = (r^2 \delta\phi')'. \quad (IV.24)$$

Introduce the function $\delta\mathcal{M} = r^2 \delta\phi'$. Eq.s(IV.23-4) then take the form

$$-\frac{\omega^2}{4\pi G} \delta\mathcal{M} = \rho \delta\mathcal{M} + r^2 \rho c \delta T' + \text{constant},$$

where the constant can only be zero, and

$$(4\pi G) r^2 (nkT^{n-1} \delta T) = \delta\mathcal{M}',$$

Elimination of δT leads to

$$-\frac{\omega^2}{4\pi G} \delta\mathcal{M} = \rho \delta\mathcal{M} + \frac{c}{4\pi G k n} r^2 \rho \left(\frac{\delta\mathcal{M}'}{x^2 T^{n-1}} \right)'.$$

Changing the scale as before we get

$$-\nu^2 \delta\mathcal{M} = f^n \delta\mathcal{M} + \frac{1}{n} x^2 f^n \left(\frac{\delta\mathcal{M}'}{x^2 f^{n-1}} \right)', \quad \nu^2 = \frac{\omega^2}{4\pi G k T^n(0)}. \quad (IV.25)$$

The crucial point is the choice of the correct boundary conditions, at $x = 0$ as well as the outer surface ($x = X$). At the center the solutions take one of two forms, $1 + Cx^2 + \dots$, which is unphysical, or else $x^3 + Cx^5 + \dots$. Accordingly we set

$$\delta\mathcal{M}(x) = x^3 g(x), \quad g(0) = 1, \quad g'(0) = 0. \quad (IV.26)$$

The boundary conditions at the outer boundary are determined by the fact that the mass is conserved,

$$\delta M = \delta\mathcal{M}(X) = 0.$$

The equations then imply that the null point is of order n . With these boundary conditions (IV.25) becomes a well defined Sturm-Liouville problem with an essentially self adjoint, second order differential operator.

Numerical calculations with the help of Mathematica are not difficult in the case of integer values of n . It is found that, when $n = 2$ and for $n = 3$, $\delta\mathcal{M}(X)$ is positive in the whole range, for all negative values of ν^2 and for positive values below a limit ν_0^2 that is about .06 for $n = 2$ and compatible with 0 for $n = 3$. The latter is the first, nodeless solution of a sequence of solutions that we have not determined in detail. The function falls to zero at the surface, where there is an n th order zero. Above this lowest value of ν^2 is a discrete set of other values of ν^2 at which the boundary condition is satisfied.

At the special value $n = 3$ the ‘ground state’, the lowest value of ν^2 , has approached very close to zero.

Polytropes with $n = 4$ are widely believed to be unstable, but a positive proof of this is not known. We have searched for harmonic solutions with negative values of ν^2 . The value $n = 4$ is indicated because it is the only integer in the interesting range, and because Mathematica is much more managable in this case. (Accuracy is lost when non integral powers of negative numbers appear at the end point.) There seems to be a discrete, decaying nodeless mode with $\nu^2 = -.015796$, but a bifurcation at this point in parameter space makes the conclusion uncertain. We carried the calculation to 15 significant figures in ν^2 but solutions do not converge towards a function that vanishes at the surface. To overcome this difficulty we reformulated the problem in terms of the variational calculus. The “solution” found for $\nu^2 = -.015796$, truncated near both ends, was used as a trial function, to show conclusively that the spectrum of ν^2 extends this far. Among many papers on this topic we mention Cowling (1936) and Ledoux (1941).

IV.5. The case $n = 3$

This case is widely believed to mark the boundary between stable and unstable polytropes. The equations are conformally invariant and a time independent solution is found by an infinitesimal conformal (homology) transformation,

$$\delta f = r f' + f. \quad (IV.26)$$

This does not represent an instability, but a “flat direction”, a perturbation from which the system does not spring back, nor does it run away. There must also be a second solution, linear in t , of the form

$$\delta f = t(r f)', \quad \delta \rho = t(r \rho' + 3\rho).$$

The equation of continuity becomes $r \rho' + 3\rho + v \rho' + r^{-2}(r^2 v)' \rho = 0$, whence $v = -r$.

This linear perturbation is the first order approximation to the exact solution found by Goldreich and Weber (1980), of the form

$$f(r, t) = \frac{1}{a(t)} \tilde{f}(x), \quad x = r/a(t).$$

The continuity equation is solved by $v = \dot{a}x$; thus $\Phi = -(\dot{a}/a)(r^2/2)$, and

$$\dot{\phi} - \vec{v}^2/2 = -a\dot{a}x^2/2 = cT + \phi.$$

This leads to

$$\tilde{\phi} = a(t)\phi \propto \tilde{f} + \kappa a^2 \ddot{a} x^2 / 6, \quad \kappa = 3k^{1/3}/c,$$

and Poisson's equation becomes

$$\tilde{f}^3 + \frac{1}{x^2}(x^2 \tilde{f}')' = \frac{-\kappa}{x^2} a^2 \ddot{a} x^2 / 6 = -\kappa a^2 \ddot{a} = \lambda, \quad \text{constant.} \quad (IV.27)$$

There is a first integral,

$$\frac{\kappa}{2} \dot{a}^2 - \lambda/a = C, \quad \text{constant.}$$

Rescaling of t and a reduces this to one of three cases

$$\dot{a} = \sqrt{1 + 1/a}, \quad \dot{a} = \sqrt{1 - 1/a}, \quad \dot{a} = 1/\sqrt{a},$$

but only the first is compatible with analyticity at $t = 0$, thus

$$t = \sqrt{a}\sqrt{1 + a} - \text{arcsinh}\sqrt{a}.$$

Setting $a = 1 + b$ we find

$$t = \sqrt{1/2}(b - b^2/2) + o(b^3)$$

The factor $a(t)$ is zero at a finite, negative value of t and increases monotonously to infinity, passing through 1 at $t = 0$. We can of course reverse the direction of flow of t to get collapse in the finite future.

Eq.(IV.27) was solved numerically (Goldreich and Weber, 1980). The solution is similar to the solution of Emden's equation, just prolonged a little at the outer end, so long as $0 < \lambda < .00654376$. For larger values of λ the distribution does not reach zero and increases for large r . For similar studies of collapsing, isothermal spheres see Hunter (1977) and references therein.

It is sure, therefore, that the polytrope with $n = 3$ is not stable. Suitably perturbed, the star may expand or collapse, until the higher or lower density causes a change in the equation of state. Among many papers on collapse we may mention Arnett (1977), Cheng (1978), Hunter (1977) and Van Riper (1978).

V. General Relativity

V.1. Lorentz invariance

The limitation to small velocities, small compared to the velocity of light, is justified almost always, with the sole exception of the photon gas. We shall now modify our treatment of the non relativistic gas of massive particles to make it consistent with relativistic invariance.

We need a 4-dimensional velocity and an associated velocity potential,

$$v_\mu = \partial_\mu \psi =: \psi_\mu, \quad \mu = 0, 1, 2, 3,$$

where ψ is a scalar field. There is only one reasonable lagrangian (Fronsdal 2007),

$$\mathcal{L} = \frac{\rho}{2}(g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - c^2) - V[\rho]. \quad (5.1)$$

The metric is the Lorentzian $g = \text{diag}(c^{-2}, -1, -1, -1)$. In the case of velocities small compared to c we set

$$\psi = c^2 t + \Phi$$

and find to order $o(c^{-2})$ the non relativistic lagrangian (1.10). Henceforth $c = 1$.

We easily allow for a dynamical gravitational field by generalizing the measure,

$$A = \int dt d^3x \sqrt{-g} \mathcal{L}.$$

In a weak, terrestrial gravitational field the usual approximation for the metric is $g = \text{diag}(1 - 2gz, -1, -1, -1)$, which leads to (2.10).

The concept of energy (density) is all-important in thermodynamics and in relativistic field theories but ill defined in General Relativity. However, as long as we limit our attention to time independent configurations, we expect to be on relatively safe grounds when we identify the energy density with the time-time component of the energy-momentum tensor,

$$T_{\mu\nu} = \rho\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\mathcal{L}. \quad (5.2)$$

In the non relativistic limit T_{00} is our hamiltonian augmented with the rest mass.

The Euler-Lagrange equations include the conservation law

$$\partial_\mu J^\mu = 0, \quad J^\mu := \sqrt{-g}g^{\mu\nu}\psi_{,\nu}.$$

The integral $\int \sqrt{-g}\rho d^3x$ is a constant of the motion (for appropriate boundary conditions) and can be interpreted as mass. This is an essential improvement over the traditional treatment. A conserved current also permits an application to a non neutral plasma (Fronsdal 2007). The (conserved) mass plays a central role in fixing the boundary conditions in the non relativistic theory; to retain this feature in the relativistic extension is natural.

V.2. Polytropic star with radiation

Here we propose to try out the lagrangian (2.7) or its relativistic version for the mixture of an ideal gas with the photon gas. In the case that the radiation pressure is relatively unimportant there is nothing new in this, and in the special case that $n = 3$ the theory is identical with that of Eddington.

In the relativistic case the action principle offers advantages even in this particular case. Clarification of the role of mass, which is confused or at least confusing in the traditional treatment, is an important part of it. Another advantage is the relative ease with which one may proceed to study mixtures.

Variation of the action with respect to the temperature gives the relation (II.10) that shows a departure from the polytropic relation $\rho = kT^n$ when $n \neq 3$. (If this last relation

is accepted, in lieu of (II.10), then from this point on the equations of motion are the same as with other methods.) The relation between Eddington's parameter β and k, n is

$$\frac{1}{\beta} = p_{\text{tot}}/p_{\text{gas}} = 1 + \frac{a}{3\mathcal{R}k};$$

It is constant only when $n = 3$. In the relativistic theory, the same relations hold; Eq.(II.10) remains valid. The equation that determines the temperature is transcendental; the first approximation is $\log k/k_0 = 3$.

Applications to real stars should await the incorporation of heat flow, not important in the case of an isolated atmosphere and of secondary importance in the case of the earthly atmosphere, but perhaps vital for a full understanding of the physics.

VI. Conclusions

VI.1. On variational principles

Variational principles have a very high reputation in most branches of physics; they even occupy a central position in classical thermodynamics, see for example the authoritative treatment by Callen (1960). An action is available for the study of laminar flows in hydrodynamics, see e.g. Fetter and Walecka (1960), though it does not seem to have been much used. Without the restriction to laminar flows it remains possible to formulate an action principle (Taub 1954, Bardeen 1970, Schutz 1970), but the proliferation of velocity potentials is confusing and no application is known to us. Recently, variational principles have been invoked in special situations that arise in gravitation.

In this paper we rely on an action principle formulation of the full set of laws that govern an ideal gas, in the presence of gravity and radiation. To keep it simple we have restricted our attention to laminar, hydrodynamical flows.

It was shown that there is an action that incorporates both of Poisson's laws as variational equations, the temperature field being treated as any other dynamical variable. The idea of varying the action with respect to the temperature is much in the classical tradition. The variational equations of motion are exactly the classical relations if radiation is neglected, or if $n = 3$.

The first encouraging result comes with the realization that the hamiltonian gives the correct expression for the internal energy and the pressure, including the contributions of radiation, under the circumstances that are considered in classical thermodynamics; that is, in equilibrium and in the absence of gravitation. This is an indication that the theory is mathematically complete, requiring no additional input from the underlying microscopic interpretation. This conclusion is reinforced by an internal derivation of a virial theorem.

Into this framework the inclusion of a gravitational field is natural. Inevitably, it leads to pressure gradients and thus also temperature gradients. If other considerations, including the heat equation, are put aside, then the theory, as it stands, predicts the persistence of a temperature gradient in an isolated system at equilibrium. The existence of a temperature gradient in an isolated thermodynamical system is anathema to tradition, and further work is required to find the way to avoid it, or to live with it. Physical considerations indicate that the answer is to be found in the phenomenon of convection. The theory in the present form can be applied when convection is not important.

A secondary but satisfying result of this work has been the application of the action principle to the study of the energy concept. Without a well defined hamiltonian it is quite impossible to attach an operative meaning to any expression for the value of the energy; it is always defined up to an additive constant, independently for each solution of the equations of motion. With a hamiltonian at our disposal we are in a position to give voice to our misgivings concerning the way that “energy” has been invoked in some branches of physics over a period of over 100 years. Though we conclude that past demonstrations of instabilities of polytropes are inconclusive, we do not suggest that the results are wrong. It is of course agreed that $n = 3$ represents an important bifurcation point.

We have insisted on the role played by the mass in fixing the boundary conditions, verified for 3 different atmospheres. The existence of a conserved current and the associated constant of the motion is especially important in the context of General Relativity where the absence of this concept casts a shadow of doubt on the choice of boundary conditions (Fronsdal 2008). Indeed it is strange that the equation of continuity, a major pillar of non-relativistic hydrodynamics, has been abandoned without protest in the popular relativistic extension. See Kippenhahn and Weigert (1990), pages 12-13.

The interaction of the ideal gas with electromagnetic fields has been discussed in a provisional manner. The transfer of entropy between the two gases is in accord with the usual treatment of each system separately.

VI.2. Suggestions

(1) It is suggested that observation of the diurnal and seasonal variations of the equation of state of the troposphere may lead to a better understanding of the role of radiation in our atmosphere. The centrifuge may also be a practical source of enlightenment. We understand that modern centrifuges are capable of producing accelerations of up to $10^6 g$. Any positive result for the temperature gradient in an isolated gas would certainly have important theoretical consequences.

(2) We suggest the use of the lagrangian (2.7), or its relativistic extension, with T treated as an independent dynamical variable and $n' = n$. Variation with respect to T yields the adiabatic relations between ρ and T , so long as the pressure of radiation is negligible, but for higher temperatures, when radiation becomes important, the effect is to increase the effective value of n' towards the ultimate limit 3, regardless of the adiabatic index n of the gas. See in this connection the discussion by Cox and Giuli (1968), page 271. In the case that $n = 3$ there is Eddington’s treatment of the mixture of an ideal gas with the photon gas. But most gas spheres have a polytropic index somewhat less than 3 and in this case the ratio $\beta = p_{\text{gas}}/p_{\text{tot}}$ may not be constant throughout the star. The lagrangian (2.7), with n identified with the adiabatic index of the gas, gives all the equations that are used to describe atmospheres, so long as radiation is insignificant. With greater radiative pressure the polytropic index of the atmosphere is affected. It is not quite constant, but nearly so, and it approaches the upper limit 3 when the radiation pressure becomes dominant. Eddington’s treatment was indicated because he used Tolman’s approach to relativistic thermodynamics, where there is room for only one density and only one pressure. Of course, all kinds of mixtures have been studied, but the equations that govern them do not supplement Tolman’s gravitational concepts in a satisfactory manner, in our opinion. Be

that as it may, it is patent that the approximation $\beta = \text{constant}$, in the works of Eddington and Chandrasekhar, is a device designed to avoid dealing with two independent gases.

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